Robust Statistics for Signal Processing
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Motivation for Robust Statistics

- Common assumptions such as Gaussianity, linearity and independence are only \textit{approximations} to reality.

\textit{Robust statistics is a body of knowledge, partly formalised into }'\textit{theories of robustness' relating to deviations from idealised assumptions in statistics [Hampel et al. (1986)].}

- Robust statistics may be separated into two distinct but related areas
  - \textbf{Robust Estimation} - A robustification of classical estimation theory (point estimation)
  - Robust Testing - A robustification of the classical theory of statistical hypothesis testing (interval estimation)
Motivation for Robust Statistics (Cont’d)

- Often engineering systems, such as in communication, are based on a parametric model where the observations are assumed to be Gaussian.
- System performance is dependent on the accuracy of this model.
  - Classical parametric statistics are optimal under exact parametric models.
  - Performance becomes more uncertain the further we are from the assumed model.
  - Incorrect model leads to a performance decrease to an uncertain level.

- Systems designed using parametric models are very sensitive to deviations from the assumed model [Hampel et al. (1986), Huber (1981)].

Solution: Robust estimation of parametric models
Motivation for Robust Statistics (Cont’d)

- In general, the aims of robust statistics are to:
  - describe (or fit a model to) the majority of the data
  - identify (and deal with) outliers or influential points

- How far away from Gaussianity are we in reality? Experience shows [Hampel et al. 1986] that
  - high quality data sets may contain up to 1% outliers,
  - low quality data sets may contain more than 10% outliers,
  - 1 – 10% outliers is common.
Example: Electricity consumption data

Half-hourly daily French electricity consumption on April 27th to June 1st, 2007

One-week differenced French load seasonal time series at 10:00, 2009
Motivation for Robust Statistics (Cont’d)

<table>
<thead>
<tr>
<th>Description</th>
<th>Nonparametric</th>
<th>Robust</th>
<th>Parametric</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model specified in terms of general properties</td>
<td>Parametric model allowing for deviations</td>
<td>Model completely specified by several parameters</td>
</tr>
<tr>
<td>Ideal Performance</td>
<td>Mediocre/Satisfactory</td>
<td>Good</td>
<td>Very Good/Excellent</td>
</tr>
<tr>
<td>Range of Validity</td>
<td>Large</td>
<td>Medium</td>
<td>Small</td>
</tr>
</tbody>
</table>

Robust is the most appropriate approach for real-life applications
Robust Estimation
Overview

- Measures of Robustness: Quantitative and Qualitative robustness
- Location Estimation
- Linear Regression Models
- Correlated Data
- Signal Processing Applications
Define the 'neighborhood’ using e.g. the $\varepsilon$ contaminated mixture model (or 'gross error model’)

$$
\mathcal{F} = \{F \mid F = (1 - \varepsilon)F_0 + \varepsilon H\},
$$

where $F_0$ is the nominal distribution and $H$ is the contaminating distribution.

Consider

1. maximum bias

$$
b(\varepsilon) = \sup_{F \in \mathcal{F}} |\Theta(F) - \Theta(F_0)|
$$

2. maximum variance

$$
\nu(\varepsilon) = \sup_{F \in \mathcal{F}} AV(F, \Theta)
$$
Then the **asymptotic breakdown point** $\varepsilon^*$ of an estimator at $F_0$ is

$$\varepsilon^*(F_0, \hat{\Theta}) = \sup\{\varepsilon \mid b(\varepsilon) < \infty\}.$$ 

Loosely speaking, it gives the limiting fraction of gross errors (outliers) the estimator can cope with (for details, see [Huber (1981) Section 1.4], [Hampel (1986) Section 2.2]).

In many cases $\varepsilon^*$ does not depend on $F_0$ and is often the same for all the usual choices for $\mathcal{F}$.

**Maximum bias curve** plots the maximum bias ($b(\varepsilon)$) of an estimator with respect to the fraction of contamination $\varepsilon$. 

Quantitative Robustness: An Example

Maximum bias curves at $F_0 = \Phi$. Beyond the BP ($\varepsilon^*$), the maximum bias is infinite.
The Influence function, introduced as influence curve [Hampel (1968,1974)], describes the effect (on an estimator \( \hat{\Theta} \)) of adding an observation of value \( x \) to a large sample; asymptotically, it is defined by

\[
IF(x, F, \Theta) = \lim_{\Delta \to 0} \frac{\Theta((1 - \Delta)F - \Delta \delta(x)) - \Theta(F)}{\Delta}
\]

where \( \delta(x) \) denotes the point mass 1 at \( x \).

Roughly speaking, it is the first derivative of a statistic at \( F \), where \( x \) plays the role of the contamination position.

It measures the normalized asymptotic bias caused by an infinitesimal contamination at point \( x \) in the observations [Hampel (1986)].
Qualitative Robustness: An Example

Influence functions of three estimators for the standard normal distribution $\Phi$
Trade-off Robustness vs. Efficiency

- **Robustness**: resistance of the estimator towards contamination quantified by: Breakdown Point $\varepsilon^*$, Maximum Bias Curve $b(\varepsilon)$, and Influence Function $IF(x, F, \Theta)$.

- **Efficiency**: the asymptotic behavior and variance of the estimator under the nominal model (clean data) quantified by $AV(F, \Theta)$ or $IF(x, F, \Theta)$.

$$AV(F, \Theta) = \int IF(x, F, \Theta)^2 dF(x)$$
Consider the model

\[ X_t = \mu + \varepsilon_t \]

under a distribution \( F \) such that \( X \sim F(x - \mu) \). We wish to estimate \( \mu \), given i.i.d \( X_t, t = 1, \ldots, n \). The Maximum likelihood estimate is

\[
\hat{\mu}_{ML} = \arg \max_\mu \sum_{t=1}^{n} \log f(x_t - \mu)
\]

\[
\Rightarrow \sum_{i=1}^{n} \psi(x_i - \hat{\mu}_{ML}) = 0 \quad \text{where} \quad \psi = f'/f
\]

- \( F \) standard Gaussian: \( \hat{\mu} \) is the sample mean
- \( F \) double Exponential: \( \hat{\mu} \) is the sample median
Example of the Effect of Outliers

Effect of outliers on the bias of the sample mean and sample median
Example: Theoretical Robustness and Efficiency of the Sample Median

- $IF(x, \Phi, Med)$ bounded $\Rightarrow$ sample median robust. Its BP $\varepsilon^* = 50\%$

- Median minimizes the maximum bias [Huber (1981)]. When $F_0 = \Phi$, efficiency of the Median is $2/\pi = 0.64 \Rightarrow$ suggests M-estimators
M-Estimator

- The MLE is asymptotically optimal (unbiased, consistent, achieves CRB) only if model is correct. If model is incorrect, performance uncertain → possibly not robust.

- Huber’s approach generalises the MLE of a parameter $\theta$ of interest for independent observations by replacing $f_{X_t}(x_t | \theta)$ by an arbitrary function $\rho(x_t, \theta)$.

\[
\arg \max_{\theta} \sum_{t=1}^{N} \log f_{X_t}(x_t | \theta) \qquad \rightarrow \qquad \arg \max_{\theta} \sum_{t=1}^{N} \rho(x_t, \theta)
\]

- This estimator is called M-estimator (ML-type estimator).
Let $\psi$ be the derivative of $\rho$, then

$$
\arg\max_\theta \sum_{t=1}^N \log f_{X_t}(x_t|\theta) \quad \rightarrow \quad \arg\max_\theta \sum_{t=1}^N \rho(x_t, \theta)
$$

leads to

$$
\sum_{t=1}^N \frac{d \log f_{X_t}(x_t|\theta)}{d\theta} = 0 \quad \rightarrow \quad \sum_{t=1}^N \psi(x_t, \theta) = 0.
$$

$\psi$ is called the score function since it 'scores' each observation $x_t$. 
Robust M-estimators

- The class of M-estimators contains in particular
  - The sample mean
  - The sample median
  - All maximum likelihood estimators
- Huber asked the following question
  \textit{How does one make an M-estimator robust?}
- Implies determining \( f_X \), or equivalently \( \psi \), so that the M-estimator is robust
- An M-estimator is qualitatively robust if and only if \( \psi \) is bounded and continuous
Robust M-estimators

Huber M-estimator is qualitatively robust ($\psi$ or IF bounded and continuous).
Linear Regression

Assume the model

\[ Y = X\beta + e \]

where \( X \) and \( e \) are independently distributed random variables. Let \( Y(t) = y_t \) and \( x'_t \) be the \( t^{th} \) row of the matrix \( X \).

**Least-Squares Estimator (LSE)**

\[ \hat{\beta} = \arg \min_{\beta} \sum_{t=1}^{n} r_t(\beta)^2 \]

\[ r_t(\beta) = y_t - x'_t\beta, \text{ being the } t^{th} \text{ residual.} \]

The LSE solves for

\[ \sum_{t=1}^{n} r_t(\hat{\beta})x_t = 0 \]
Effect of Outliers

The regression line is tilted by the outliers

LSE without outliers

bias

LSE

x-outlier

y-outlier
M-Estimation for Regression

M-Estimator:

\[
\hat{\beta} = \arg \min_{\beta} \sum_{t=1}^{n} \rho \left( \frac{r_t(\beta)}{\hat{\sigma}_r} \right) \Rightarrow \sum_{t=1}^{n} \psi \left( \frac{r_t(\hat{\beta})}{\hat{\sigma}_r} \right) x_t = 0
\]

A leverage point (\( \|x_t\| \) large) will dominate the equation. Here, \( \hat{\sigma}_r \) is a robust scale of the residuals \( r_t \).

Solution:

- Redescending function \( \psi \): MM estimator
- Down-weight leverage points: GM estimator
- Other robust estimators are: Residual Autocovariance (RA-), Least Median Squares (LMS), Least Trimmed Squares (LTS), S-, \( \tau \)-Estimators.
LSE and M-Estimator are not qualitatively robust
Robust Methods for Correlated Data

- Difficulty introduced by the correlation
- Several types of outliers: Additive, Innovative, patchy, isolated, ...
- Different definitions of robustness measures
- Estimators need to be adapted to deal with correlation

Only a few methods exist

- 2 Definitions of influence function in the correlated data case ([Künsch (1981)] and [Martin (1981)])
- Breakdown Point ($\varepsilon^*$) still not clearly defined [Genton (2010)]
Some Robust Estimators for ARMA Models

- CML: Cleaned Maximum Likelihood after ’3-σ’ rejection (practical engineering method) ⇒ it neglects the correlation, bad performance.
- Generalized-M (GM): used in Power Systems [Maronna et al. (2006)], not robust for ARMA; for AR($p$), BP ($\varepsilon^*$) decreases with increasing $p$.
- Residual Autocovariance (RA) and truncated RA estimators (TRA) [Bustos and Yohai (1986)]. RA is not robust and TRA lacks efficiency for ARMA.
- Filtered-$\tau$ [Maronna (2006)], Ratios of Medians (RME), Medians of Ratios (MRE) and Filtered Hellinger based estimator [Chakhchoukh (2010)].
Importance of Robust Filtering

For an AR($p$), a residual $r_t$ is evaluated by regressing $y_t$ on the $p$ variables $y_{t-1}, \ldots, y_{t-p}$. An observation $y_t$ is used in computing $p+1$ residuals: $r_t, \ldots, r_{t+p}$ ⇒ BP ($\varepsilon^*$) decreases significantly.

For an ARMA model, one observation will affect all the residuals ⇒ BP $\varepsilon^* = 0$.

**Problem:** Propagation of outliers in the data used in the estimation

**Solution:** use robust residuals computed by a robust filter cleaner [Masreliez (2010)]:

$$\tilde{r}_t = y_t - \phi_1\hat{y}_{t-1|t-1} - \cdots - \phi_p\hat{y}_{t-p|t-1}$$

Cleaning the data with a robust filter improves efficiency
Robust Filtering of an AR(1)

For an AR(1): $X_t = \phi X_{t-1} + \varepsilon_t$, we use the following robust filtering algorithm:

- estimate robustly $\hat{\phi}$, e.g.: $\hat{\phi} = \frac{\hat{C}_r(1)}{\hat{C}_r(0)}$ and run the recursions of the filter-cleaner; $\hat{X}_{1|1} = X_1$, $P_{1|1} = \text{MADN}(X_t)$,

Prediction :

$$\hat{X}_{2|1} = \hat{\phi} \hat{X}_{1|1}$$
$$\hat{\varepsilon}_2 = Y_2 - \hat{\phi} \hat{X}_{1|1}$$
$$P_{2|1} = \hat{\phi}^2 P_{1|1} + \sigma_{\varepsilon}^2$$

Correction :

$$\hat{X}_{2|2} = \hat{X}_{2|1} + \frac{1}{\sqrt{P_{2|1}}} P_{2|1} \psi \left( \frac{\hat{\varepsilon}_2}{\sqrt{P_{2|1}}} \right)$$
$$P_{2|2} = P_{2|1} - \frac{1}{P_{2|1}} P_{2|1}^2 w \left( \frac{\hat{\varepsilon}_2}{\sqrt{P_{2|1}}} \right)$$

Go to the next step of the recursion.

- Apply the ML to the filtered series $\{\hat{X}_{t|t}\}$ or:
- Test to remove the outliers, e.g.: If $\hat{\varepsilon}_2 > 3 \sqrt{P_{2|1}}$ then $X_2$ is outlying
- Apply ML estimator that handles missing data.
Applications

- Wireless Communications: Robust Geolocation, eg: [Hammes (2009)]

- Array Processing: Robust Direction of Arrival Estimation, eg: [Tsakalides (1995)]
Geolocation refers to identifying the position of a mobile terminal using a network of sensors.

Applications for Geolocation arise e.g. in emergency call services, yellow page services and intelligent transport systems [Caffery (1999)].

We consider wireless positioning of a stationary terminal based on TOA estimates.

At least three sensors/BSs are needed to solve ambiguities.

- **line-of-sight (LOS)**
- **non-line-of-sight (NLOS)**
Problem Statement

- Time of arrival (TOA) estimates are multiplied by the speed of light to obtain the measured distances

\[ r_m = \sqrt{(x_m - x)^2 + (y - y_m)^2} + \tilde{v}_m, \quad m = 1, \ldots, M, \]

where \( x_m, y_m \) are the known coordinates of the BS and \( x, y \) describe the unknown location of the MT. The i.i.d. random variables \( \tilde{v}_m \) have pdf

\[ f_{\tilde{V}}(\tilde{v}) = (1 - \varepsilon)N(\tilde{v}; 0, \sigma_G) + \varepsilonH, \]

describing sensor noise and errors due to NLOS propagation (\( H = f_G \ast f_\eta \)) where \( f_\eta \) may be any pdf with positive mean such that \( E\{H\} > 0 \).
Linearization

- Squaring the nonlinear equation yields
\[
r_m^2 = h_m^2 + 2h_m\tilde{v}_m + \tilde{v}_m^2 = \nu_m(h_m)
\]

- For \(M\) BSs we have
\[
r = S\theta + \nu,
\]
where \(\theta = [x \ y \ R^2]^T\) with \(R^2 = x^2 + y^2\).

- Since \(f_V(\nu)\) is non-Gaussian and contains outliers due to NLOS, least-squares estimation suffers from a performance loss.

\[\downarrow\]
Robust methods
Iterative Robust Algorithm

1. **Initialisation:** Set \( i = 0 \). Obtain an initial estimate of \( \theta \), \( \hat{\theta}^0 \).

2. **Determine residuals:** \( \hat{v} = r - S\hat{\theta}^i \).

3. **Estimate \( \lambda \), perform transformation KDE.**

4. **Estimate score function:** \( \hat{\phi} = -\frac{f'_V(v)}{f_V(v)} \).

5. **Update:** \( \hat{\theta}^{i+1} = \hat{\theta}^i + \mu (S^T S)^{-1} S^T \hat{\phi}(\hat{v}) \) or \( (S^T \Omega S)^{-1} S^T \Omega r \), \( \Omega = \text{diag}(\omega), \ \omega = |\hat{\phi}(\hat{v})/\hat{v}| \)

6. **Check for convergence:** If \( \frac{\|\hat{\theta}_{i+1} - \hat{\theta}_i\|}{\|\hat{\theta}_{i+1}\|} < \xi \) stop, otherwise set \( i \rightarrow i + 1 \) and go to step 2.
Simulation Settings

- Consider 10 BSs each of them collecting 10 measurements.
- We compare least-squares ('LS') with Huber’s M-estimator ('\( H_c \)') where the clipping point \( c = 0.6 \hat{\sigma}_V \), where \( \hat{\sigma}_V \) is estimated using the median absolute deviation.
- Semi-parametric estimators using Newton-Raphson algorithm labeled as 'SPMR' and the one based on weighted least-squares is 'SPWLS'.
- We average over \( MC = 10,000 \) Monte-Carlo runs and \( \sigma_G = 150 \text{m} \).
- Performance measure is the mean error distance, i.e.,

\[
MED = \frac{1}{MC} \sum_{i=1}^{MC} \sqrt{(x - \hat{x}_i)^2 + (y - \hat{y}_i)^2}
\]
Simulation results

MED vs. degree of NLOS contamination. $f_\eta$ is an exponential distribution with

$$\sigma_\eta = 409 m$$
Direction of Arrival (DOA) estimates are needed in array processing
  - Smart Antennas
  - Space-Time Adaptive Processing
  - Radar

Classical methods for DOA estimation based on sample covariance matrix are not robust
  - Beamformer
  - Capon’s minimum variance
  - ML techniques
  - Subspace methods MUSIC, ESPIRIT
Robust DOA methods exist based on:

- M-Estimation
- Symmetric-Alpha-Stable (SaS) distributions and Fractional Lower Order Moments (FLOMs)
- Gaussian mixture distributions and Space Alternating Generalised Expectation Maximisation (SAGE)
- Nonparametric statistics using the spatial sign function

Former three robust DOA estimators require knowledge of noise parameters/setting of thresholds/choice of weighting functions.

Last nonparametric estimator is simple and requires no prior knowledge or settings to be chosen.
Problem: Estimate the direction-of-arrivals of the sources using the observations of a sensor array in an impulsive noise environment.
Array Signal Model

Model

\[ y_n = As_n + x_n, \quad n = 1, \ldots, N \]

\( y_n \): \( p \)-dim snapshot from \( p \) array elements
\( A = (a(\theta_1), \ldots, a(\theta_q)) \): \( p \times q \)-dim array steering matrix
\( a(\theta) \): \( p \)-dim array steering vector
\( \theta_1, \ldots, \theta_q \): directions to the \( q \) sources
\( s_n \): \( q \)-dim element source signal
\( x_n \): \( p \)-dim spherically symmetric noise

Assumptions

- snapshots \( y_n, \ n = 1, \ldots, N \), are i.i.d.
- source signal \( s_n \) and noise \( x_m \) are independent for all \( n, m = 1, \ldots, N \)
- \( q < p \) to avoid identifiability problems
- \( A \) is of full rank \( q \)
Conventional DOA estimation: MUSIC

The spatial covariance matrix has the following structure

$$ R = A R_s A^H + \sigma^2 I $$

(1)

From the eigendecomposition of $R$

$$ R = U \Sigma V $$

(2)

where $U = [u_1, u_2, \ldots, u_M]$ and $\Sigma = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_M]$.

- Construct the signal and the noise subspace, $U_s$ and $U_n$, respectively
- Search for the peaks in the MUSIC pseudo-Spectrum $P(\theta)$

$$ P(\theta) = \frac{1}{\| U_n^H a(\theta) \|^2} $$

(3)


Conventional DOA Estimation

The estimated covariance matrix is given by

\[
\hat{R} = \frac{1}{N} \sum_{t=1}^{N} xx^H
\]

\[\Downarrow\]

Robust Estimation:

- Normalize each of the snapshots, called spatial sign function
- Trim the corrupted observations, requires hypothesis testing
- Robust estimation of spatial covariance matrix, e.g. FLOM, maximum likelihood estimation. Some possible methods for robust covariance estimation MCD, MVE, MM-, ...
Spatial Sign Function

- Spatial sign function (SSF) of a $p$-variate complex vector $\mathbf{x}$

\[
\mathbf{u}(\mathbf{x}) = \begin{cases} 
\frac{\mathbf{x}}{||\mathbf{x}||} & \mathbf{x} \neq 0 \\
0 & \mathbf{x} = 0
\end{cases}
\]

- $\mathbf{u}$ is a unit length direction vector
- Generalises the sign function $\text{sgn}(\mathbf{x})$ for 1-D to $p$-D
Sample spatial sign covariance matrix (SCM) of

$$R_1 = E[u(x)u^H(x)]$$

$$\hat{R}_1 = \frac{1}{N} \sum_{n=1}^{N} u(x_n)u^H(x_n)$$

also known as quadrant correlation

Sample spatial tau covariance matrix (TCM) of

$$R_2 = E[u(x - y)u^H(x - y)]$$

$$\hat{R}_2 = \frac{1}{N(N - 1)} \sum_{n=1}^{N} \sum_{m=1}^{N} u(x_n - x_m)u^H(x_n - x_m)$$
Simulation results

Setup

- Two linear FM signals impinging on an array of \( m = 8 \) sensors in ULA geometry
- DOAs are \([-3^\circ, 2^\circ]\)
- Total number of snapshots \( N = 128 \)
- \( \epsilon \)-contaminated mixture, \( \epsilon = 0.2 \) and \( \kappa = 20 \)

RMSE DOA Estimation
Conclusions

- Robust methods are useful tools for estimation in many real world applications.
- Robust statistics for independently and identically distributed data are well-established.
- There exists a need for robust techniques for correlated data: the more interesting case for a signal processing practitioner.
- Optimality has its advantage, but Robustness is the engineer’s interest.
Robust Statistics for Signal Processing

References
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