

# Direct Design of Orthogonal Filter Banks and Wavelets by Sequential Convex Quadratic Programming

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September 2008

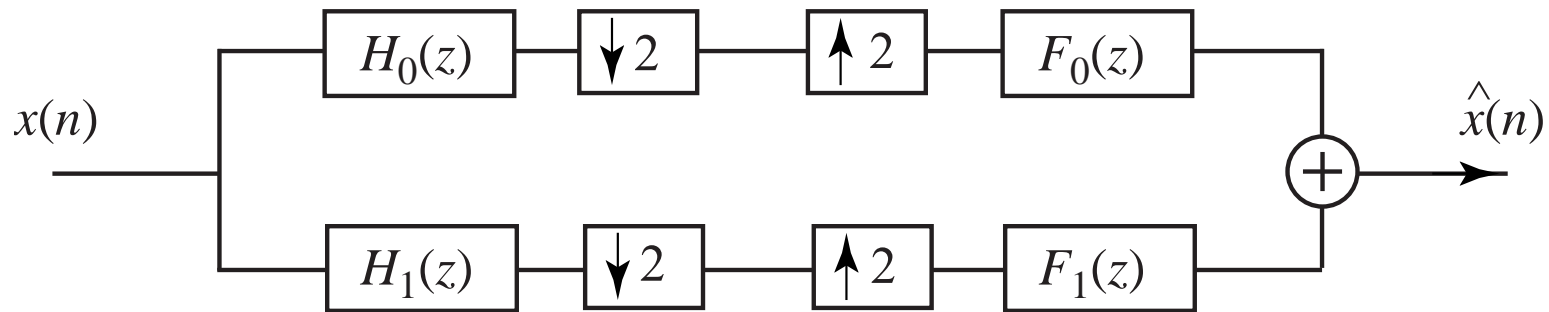
Abstract Two-channel conjugate quadrature (CQ) FIR filter banks are among the most popular building blocks for multirate systems as they offer precise perfect reconstruction property. Most design methods for CQ filters are indirect in that a halfband filter with nonnegativity constraint is designed, followed by a spectrum factorization. This talk describes a direct method that does not require designing the halfband filter and its factorization, and integrates least-square and minimax designs with vanishing moment and other requirements into a single design framework. Examples are supplied to help examine design performance, efficiency, and its ability of getting global solutions.

# Outline

- Introduction
- Early and recent work
- Constrained linear updates and a convex QP formulation for least-squares design of conjugate quadrature (CQ) filters
- Constrained linear updates and an second-order cone programming (SOCP) formulation for minimax (equiripple) design of CQ filters
- Experimental results

# 1. Introduction

- Two-channel FIR filter bank



$$\hat{X}(z) = \frac{1}{2}[F_0(z)H_0(z) + F_1(z)H_1(z)]X(z) + \frac{1}{2}[F_0(z)H_0(-z) + F_1(z)H_1(-z)]X(-z)$$

- Perfect reconstruction (PR) conditions

$$F_0(z)H_0(z) + F_1(z)H_1(z) = 2z^{-l}$$

$$F_0(z)H_0(-z) + F_1(z)H_1(-z) = 0$$

- A conjugate quadrature (CQ) filter bank assumes

$$H_1(z) = -z^{-(N-1)}H_0(-z^{-1}), \quad F_0(z) = z^{-(N-1)}H_0(z^{-1}), \quad F_1(z) = z^{-(N-1)}H_1(z^{-1})$$

⇒ the 2<sup>nd</sup> PR condition is automatically satisfied

(no aliasing) and the 1<sup>st</sup> PR condition becomes

$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2 \quad (\text{PS})$$

which is called the *power symmetric (PS) condition*

because it implies

$$\left| H_0\left(e^{j(\pi/2-\theta)}\right) \right|^2 + \left| H_0\left(e^{j(\pi/2+\theta)}\right) \right|^2 = 1 \quad \text{for any } \theta$$

## 2. Early and Recent Work

- Representative early and recent work include
  - Smith and Barnwell (1984)
  - Mintzer (1985)
  - Vaidyanathan and Nguyen (1987)
  - Rioul and Duhamel (1994)
  - Lawton and Michelli (1997)
  - Tuqan and Vaidyanathan (1998)
  - Dumitrescu and Popeea (2000)
  - Tay (2005, 2006)

- The most common design technique:
  - ◆ A half-band filter  $P(z)$  is a zero-phase FIR filter satisfying

$$P(z) + P(-z) = 2$$

- ◆ If we define  $P(z) = H_0(z)H_0(z^{-1})$ , then the PS condition

$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2$$

becomes

$$P(z) + P(-z) = 2$$

So  $P(z)$  is a half-band filter and, it is *nonnegative*

*everywhere*:  $P(e^{j\omega}) = H_0(e^{j\omega})H_0(e^{-j\omega}) = |H_0(e^{j\omega})|^2 \geq 0 \quad (\mathbf{P})$

◆ Design steps:

- (a) Design a lowpass half-band FIR filter  $P(z)$  with nonnegativity property  $P(e^{j\omega}) \geq 0$
- (b) Perform a spectral decomposition

$$P(z) = H_0(z)H_0(z^{-1})$$

- Vanishing moment (VM): the number of VMs equals to the number of zeros of  $H_0$  at  $\omega = \pi$ :

$$\left. \frac{d^l H_0(e^{j\omega})}{d\omega^l} \right|_{\omega=\pi} = (-j)^l \sum_{n=0}^{N-1} (-1)^n n^l h_n = 0, \quad \text{for } l = 0, 1, \dots, L-1$$

### 3. Least-Squares Design of CQ Filters

#### Problem Formulation

- Let

$$H_0(z) = \sum_{n=0}^{N-1} h_n z^{-n} \text{ with } N \text{ even, and } h = [h_0 \ h_1 \ \dots \ h_{N-1}]^T$$

- A direct approach: minimizing a least squares type objective function subject to the PS constraint:

$$\underset{h}{\text{minimize}} \quad \int_{\omega_a}^{\pi} |H_0(e^{j\omega})|^2 d\omega$$

$$\text{subject to: } H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2$$

- The objective function is a positive definite quadratic form:

$$\int_{\omega_a}^{\pi} |H_0(e^{j\omega})|^2 d\omega = h^T Q h$$

where  $Q$  is a symmetric positive definite Toeplitz matrix:

$$Q = \text{toeplitz} \left( \left[ \begin{array}{cccc} \pi - \omega_a & -\sin \omega_a & \cdots & \frac{-1}{N-1} \sin[(N-1)\omega_a] \end{array} \right] \right)$$

- The constraint is the PS condition:

$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2 \quad (\text{PS})$$

which is equivalent to a set of  $N/2$  second-order equality

constraints: 
$$\sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m) \quad \text{for } m = 0, 1, \dots, (N-2)/2$$

where  $\delta(m) = 1$  for  $m = 0$  and  $\delta(m) = 0$  for  $m \neq 0$ .

- The design problem now becomes a *polynomial optimization problem* (POP):

$$\underset{h}{\text{minimize}} \quad h^T Q h = \|Q^{1/2} h\|^2$$

$$\text{subject to:} \quad \sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m) \quad \text{for } 0 \leq m \leq (N-2)/2$$

- The POP can be modified to include VM requirement:

$$\underset{h}{\text{minimize}} \quad h^T Q h = \|Q^{1/2} h\|^2$$

$$\text{subject to:} \quad \sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m) \quad \text{for } 0 \leq m \leq (N-2)/2$$

$$\sum_{n=0}^{N-1} (-1)^n n^l h_n = 0 \quad \text{for } l = 0, 1, \dots, L-1$$

- The POP can be further modified to reduce the filter's phase response nonlinearity

$$\begin{aligned}
 & \underset{h}{\text{minimize}} && \left\| Q^{1/2} h \right\|^2 + w \left\| \begin{bmatrix} I_{N_1} & -\tilde{I}_{N_1} \end{bmatrix} h \right\|^2 \\
 & \text{subject to:} && \sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m) \quad \text{for } 0 \leq m \leq (N-2)/2 \\
 & && \sum_{n=0}^{N-1} (-1)^n n^l h_n = 0 \quad \text{for } l = 0, 1, \dots, L-1
 \end{aligned}$$

- Features of these problems:
    - ◆ All polynomials are of second-order.
    - ◆ The objective function is convex
    - ◆ These are *nonconvex* problems because of the  $N/2$  second-order equality constraints (the PS conditions).
  - The PS constraints: examples
1.  $N = 4 \Rightarrow 2$  constraints:

$$h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1$$

$$h_0 \cdot h_2 + h_1 \cdot h_3 = 0$$

## 2. $N = 20 \Rightarrow 10$ constraints:

$$h_0^2 + h_1^2 + \cdots + h_{19}^2 = 1 \quad (20 \text{ terms})$$

$$h_0 \cdot h_2 + h_1 \cdot h_3 + \cdots + h_{17} \cdot h_{19} = 0 \quad (18 \text{ terms})$$

$$h_0 \cdot h_4 + h_1 \cdot h_5 + \cdots + h_{15} \cdot h_{19} = 0 \quad (16 \text{ terms})$$

$$h_0 \cdot h_6 + h_1 \cdot h_7 + \cdots + h_{13} \cdot h_{19} = 0 \quad (14 \text{ terms})$$

$\vdots$

$\vdots$

$$h_0 \cdot h_{16} + h_1 \cdot h_{17} + h_2 \cdot h_{18} + h_3 \cdot h_{19} = 0 \quad (4 \text{ terms})$$

$$h_0 \cdot h_{18} + h_1 \cdot h_{19} = 0 \quad (2 \text{ terms})$$

## Constrained Linear Updates

- Suppose we are in the  $k$ th iteration of the algorithm and we want to update filter coefficients

$$\text{from } h^{(k)} \text{ to } h^{(k+1)} = h^{(k)} + d$$

to achieve two things:

- ◆ to reduce the filter's stopband energy  $h^T Q h$
- ◆ to better satisfy constraints

$$\sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m), \quad 0 \leq m \leq N/2 - 1$$

- The stopband energy at  $h^{(k+1)}$  is equal to  $\|Q^{1/2}(d + h^{(k)})\|^2$
- The constraints at  $h^{(k+1)}$  becomes

$$\sum_n h_n^{(k)} h_{n+2m}^{(k)} + \sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)} + \sum_n d_n d_{n+2m} = \delta(m)$$

- Imposing constraints on the smallness of increment vector  $d$ :

$$|d_i| \leq \beta \quad \text{for } i = 1, 2, \dots, N$$

then the 2<sup>nd</sup>-order constraints can be linearized:

$$\begin{aligned} & \sum_n h_n^{(k)} h_{n+2m}^{(k)} + \sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)} + \underbrace{\sum_n d_n d_{n+2m}}_{\text{very small, drop}} \\ & \approx \underbrace{\sum_n h_n^{(k)} h_{n+2m}^{(k)}}_{\text{a known term, denoted by } s^{(k)}(m)} + \underbrace{\sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)}}_{\text{linear updates}} \\ & = \delta(m) \quad \text{for } m = 0, 1, \dots, (N-2)/2 \end{aligned}$$

- This leads to a set of  $(N - 2)/2$  linear equations:

$$\sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)} = \delta(m) - s^{(k)}(m) \equiv u^{(k)}(m)$$

which can in matrix-vector notation be expressed as

$$C^{(k)} d = u^{(k)}$$

- The smallness constraint on  $d$  is given by

$$|d_i| \leq \beta \quad \text{for } 1 \leq i \leq n \quad \Leftrightarrow \quad Ad \leq b$$

- The linear constraint on VMs is given by

$$\sum_{n=0}^{N-1} (-1)^n n^l (d_n + h_n^{(k)}) = 0 \quad \text{for } 0 \leq l \leq L-1 \quad \Leftrightarrow \quad Dd = v^{(k)}$$

## A Quadratic Programming (QP) Formulation

- Summarizing the analysis, the solution strategy is to *iteratively* update the filter coefficients from  $h^{(k)}$  to  $h^{(k+1)} = h^{(k)} + d^{(k)}$  where  $d^{(k)}$  is obtained by solving the QP problem

$$\underset{d}{\text{minimize}} \quad \left\| Q^{1/2} (d + h^{(k)}) \right\|^2$$

$$\text{subject to: } Ad \leq b$$

$$\begin{bmatrix} C^{(k)} \\ D \end{bmatrix} d = \begin{bmatrix} u^{(k)} \\ v^{(k)} \end{bmatrix}$$

- If the phase-response nonlinearity is also a concern, we update  $h^{(k)}$  to  $h^{(k+1)} = h^{(k)} + d^{(k)}$  by solving the QP problem

$$\underset{d}{\text{minimize}} \quad \left\| Q^{1/2} (d + h^{(k)}) \right\|^2 + w \left\| \begin{bmatrix} I_{N_1} & -\tilde{I}_{N_1} \end{bmatrix} (d + h^{(k)}) \right\|^2$$

$$\text{subject to:} \quad Ad \leq b$$

$$\begin{bmatrix} C^{(k)} \\ D \end{bmatrix} d = \begin{bmatrix} u^{(k)} \\ v^{(k)} \end{bmatrix}$$

## 4. Minimax Design of CQ Filters

### Problem Formulation

- The formulation in this case is changed to

$$\begin{aligned} & \underset{h}{\text{minimize}} \underset{\omega_a \leq \omega \leq \pi}{\text{maximize}} \quad \left| H_0(e^{j\omega}) \right| \\ \text{subject to:} \quad & \sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m) \quad \text{for } 0 \leq m \leq (N-2)/2 \\ & \sum_{n=0}^{N-1} (-1)^n n^l h_n = 0 \quad \text{for } l = 0, 1, \dots, L-1 \end{aligned}$$

## Constrained Linear Updates

- Like in the least squares design, the constrained linear update gives

$$\underset{d}{\text{minimize}} \quad \underset{\omega_a \leq \omega \leq \pi}{\text{maximize}} \quad |H_0(e^{j\omega})|$$

$$\text{subject to:} \quad Ad \leq b$$

$$\begin{bmatrix} C^{(k)} \\ D \end{bmatrix} d = \begin{bmatrix} u^{(k)} \\ v^{(k)} \end{bmatrix}$$

- Dealing with the objective function, we write

$$H_0(e^{j\omega}) = \sum_{n=0}^{N-1} h_n e^{-jn\omega} = h^T c(\omega) - jh^T s(\omega)$$

$$c(\omega) = [1 \quad \cos \omega \quad \cdots \quad \cos(N-1)\omega]^T, \quad s(\omega) = [0 \quad \sin \omega \quad \cdots \quad \sin(N-1)\omega]^T$$

hence

$$|H_0(e^{j\omega})| = \sqrt{(h^T c(\omega))^2 + (h^T s(\omega))^2} = \left\| \begin{bmatrix} c(\omega)^T \\ s(\omega)^T \end{bmatrix} \cdot h \right\| \equiv \|F(\omega) \cdot h\|$$

$$\Rightarrow |H_0(e^{j\omega}, h^{(k)} + d^{(k)})| = \|F(\omega) \cdot (h^{(k)} + d^{(k)})\| = \|F(\omega)d^{(k)} + g^{(k)}\|$$

- This converts the minimax problem into

minimize  $\eta$

subject to:  $\|F(\omega_i)d^{(k)} + g^{(k)}\| \leq \eta$  for  $\{\omega_i\} \subseteq [\omega_a, \pi]$

$$Ad \leq b$$

$$\begin{bmatrix} C^{(k)} \\ D \end{bmatrix} d = \begin{bmatrix} u^{(k)} \\ v^{(k)} \end{bmatrix}$$

which is an SOCP problem.

## 5. Experimental Results

- We performed a total of 18 LS designs with (filter length, stopband edge) being  $(N = 32, \omega_a = 0.58\pi)$ ,  $(N = 64, \omega_a = 0.57\pi)$ , and  $(N = 96, \omega_a = 0.56\pi)$ , and the number of vanishing moments  $L$  from 0, 1, ..., to 5. The associated QP were solved by the Optimization Toolbox provided by MathWorks Inc.
- The design performance was evaluated in terms of stopband energy and satisfaction of the PS conditions.

Table 1: Least squares with  $N = 32$ ,  $\omega_a = 0.58\pi$

$L$	Energy in Stopband	Largest Equation Error
0	$2.4470 \times 10^{-5}$	$5 \times 10^{-16}$
1	$2.5260 \times 10^{-5}$	$5 \times 10^{-16}$
2	$2.5260 \times 10^{-5}$	$5 \times 10^{-16}$
3	$2.9658 \times 10^{-5}$	$5 \times 10^{-16}$
4	$2.9658 \times 10^{-5}$	$5 \times 10^{-16}$
5	$3.9914 \times 10^{-5}$	$5 \times 10^{-16}$

LS design with  $N = 32$ ,  $\omega_a = 0.58\pi$ , and  $L = 3$

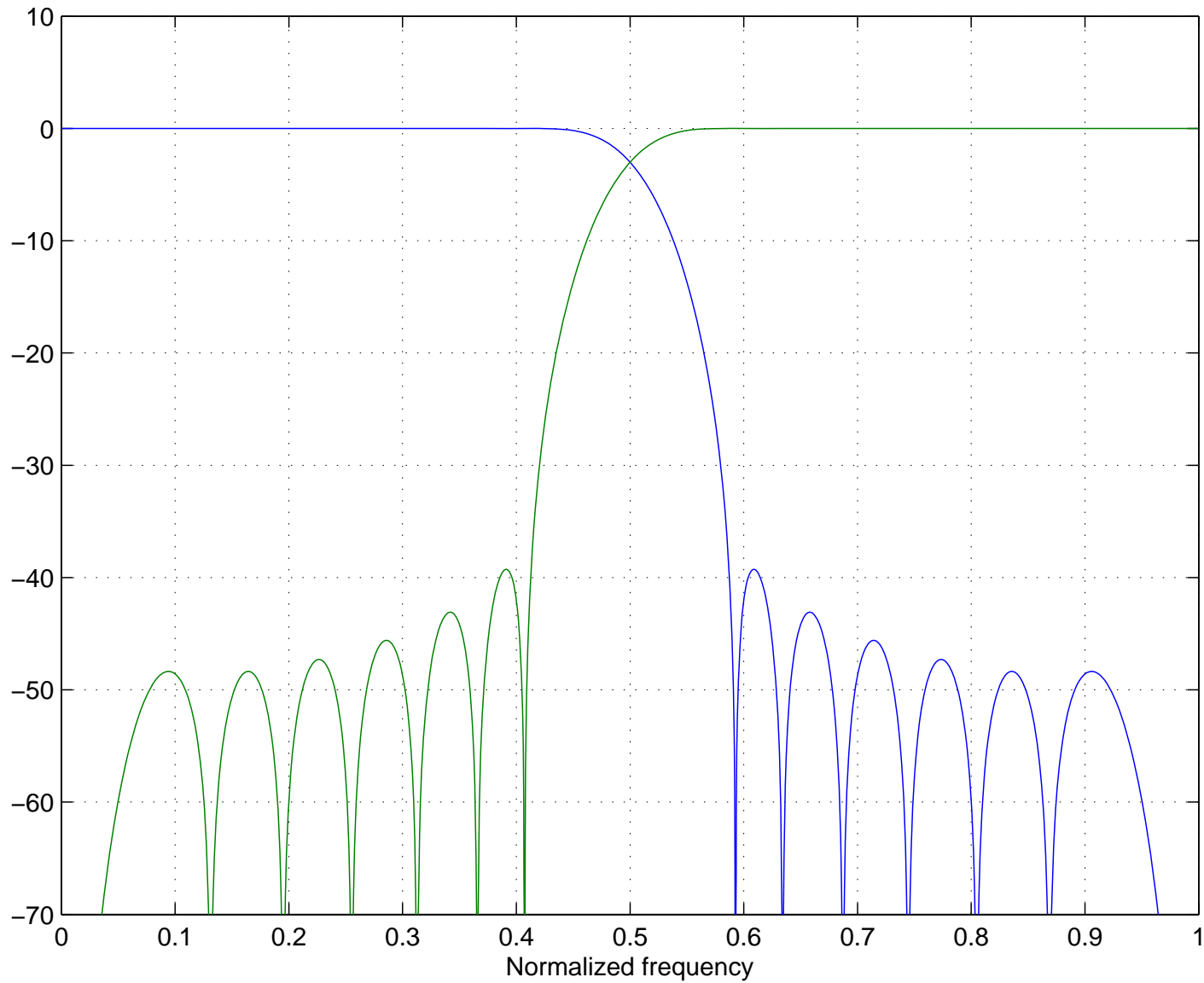


Table 2: Least squares with  $N = 64$ ,  $\omega_a = 0.57\pi$

$L$	Energy in Stopband	Largest Equation Error
0	$3.9962 \times 10^{-8}$	$5 \times 10^{-16}$
<b>1</b>	<b><math>4.0215 \times 10^{-8}</math></b>	<b><math>5 \times 10^{-16}</math></b>
2	$4.0215 \times 10^{-8}$	$5 \times 10^{-16}$
3	$4.3110 \times 10^{-8}$	$5 \times 10^{-16}$
4	$4.3110 \times 10^{-8}$	$5 \times 10^{-16}$
5	$4.8897 \times 10^{-8}$	$5 \times 10^{-16}$

LS design with  $N = 64$ ,  $\omega_a = 0.57\pi$ , and  $L = 1$

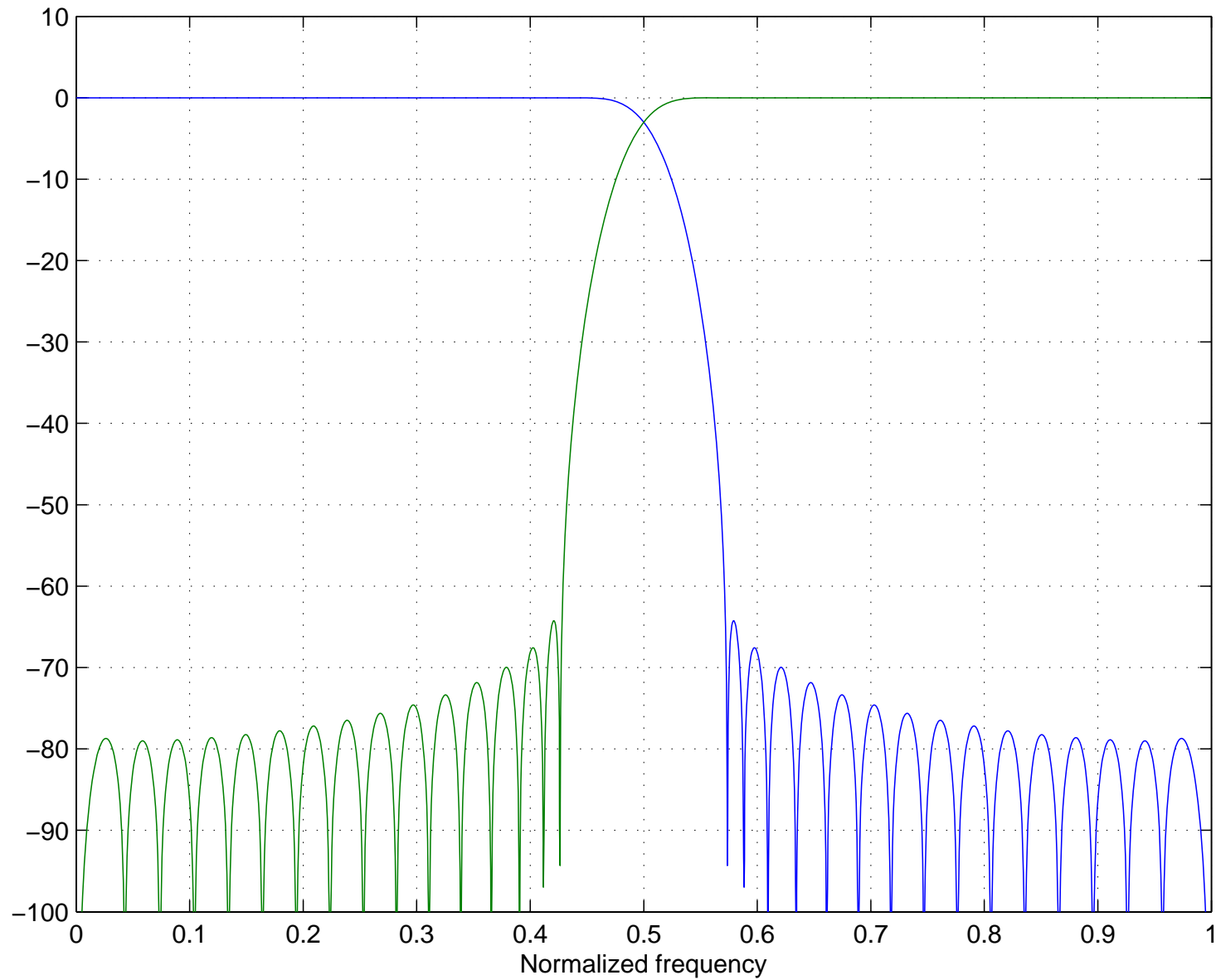
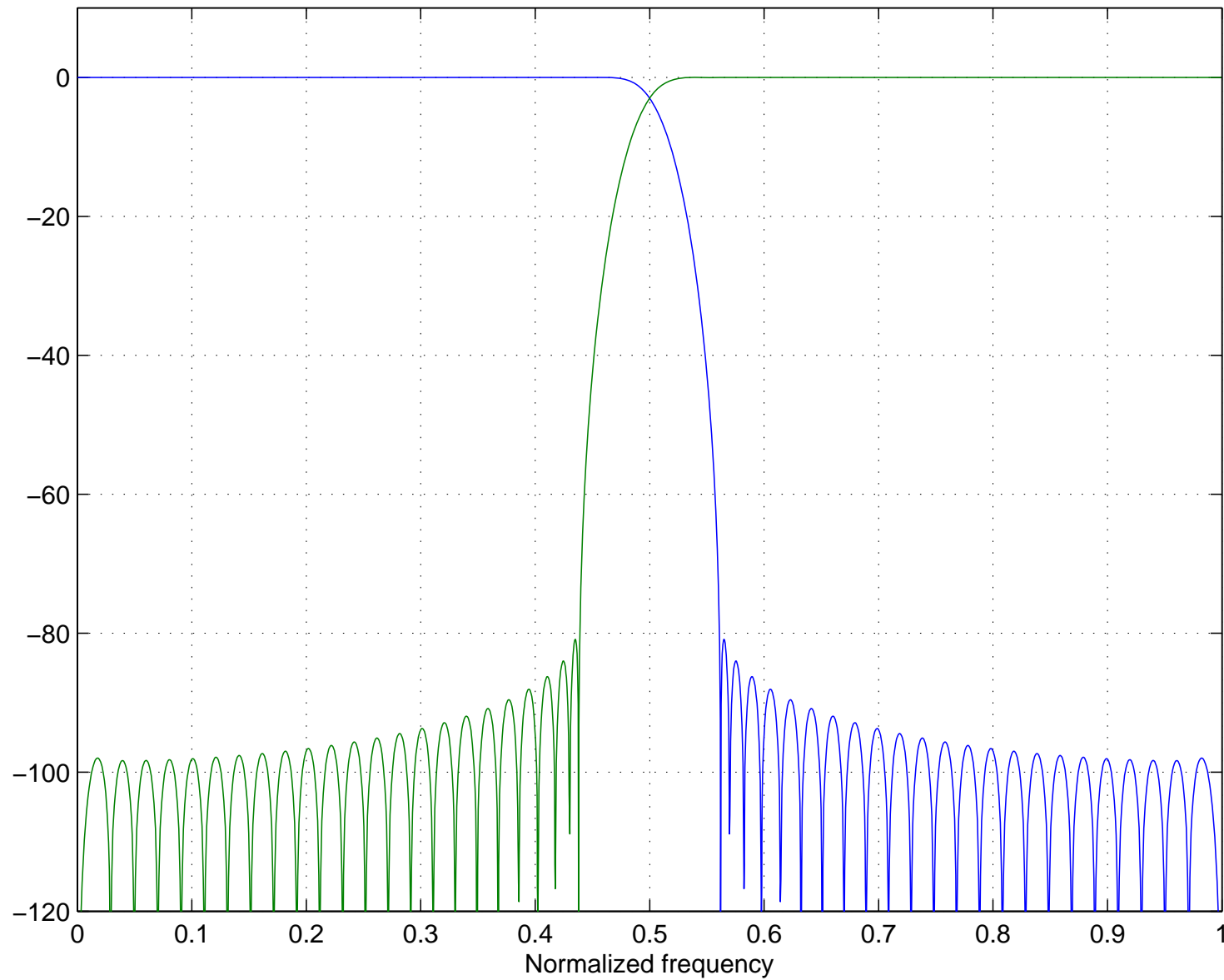


Table 3: Least squares with  $N = 96$ ,  $\omega_a = 0.56\pi$

$L$	Energy in Stopband	Largest Equation Error
0	$5.6213 \times 10^{-10}$	$5 \times 10^{-16}$
1	$5.6660 \times 10^{-10}$	$5 \times 10^{-16}$
2	$5.6660 \times 10^{-10}$	$5 \times 10^{-16}$
3	$5.8954 \times 10^{-10}$	$5 \times 10^{-16}$
4	$5.8954 \times 10^{-10}$	$5 \times 10^{-16}$
5	$6.2901 \times 10^{-10}$	$7.6190 \times 10^{-10}$

LS design with  $N = 96$ ,  $\omega_a = 0.56\pi$ , and  $L = 2$



- We also performed a total of 18 minimax designs with (filter length, stopband edge) being  $(N = 32, \omega_a = 0.58\pi)$ ,  $(N = 64, \omega_a = 0.57\pi)$ , and  $(N = 96, \omega_a = 0.56\pi)$ , and the number of vanishing moments  $L$  from 0, 1, ..., to 5. The associated SOCP problems were solved by SeDuMi 1.1R2 (a freeware maintained by the Advanced Optimization Laboratory at McMaster).
- The design performance was evaluated in terms of instantaneous stopband energy and satisfaction of the PS conditions.

Table 4: Minimax with  $N = 32$ ,  $\omega_a = 0.58\pi$

$L$	Instantaneous Energy in Stopband	Largest Equation Error
0	$1.0308 \times 10^{-4}$	$5 \times 10^{-16}$
1	$1.0878 \times 10^{-4}$	$4.0666 \times 10^{-6}$
2	$1.1460 \times 10^{-4}$	$3.8830 \times 10^{-6}$
3	$1.3232 \times 10^{-4}$	$5.0572 \times 10^{-6}$
4	$1.3183 \times 10^{-4}$	$2.6276 \times 10^{-7}$
5	$1.9008 \times 10^{-4}$	$2.5628 \times 10^{-7}$

Minimax design with  $N = 32$ ,  $\omega_a = 0.58\pi$ , and  $L = 4$

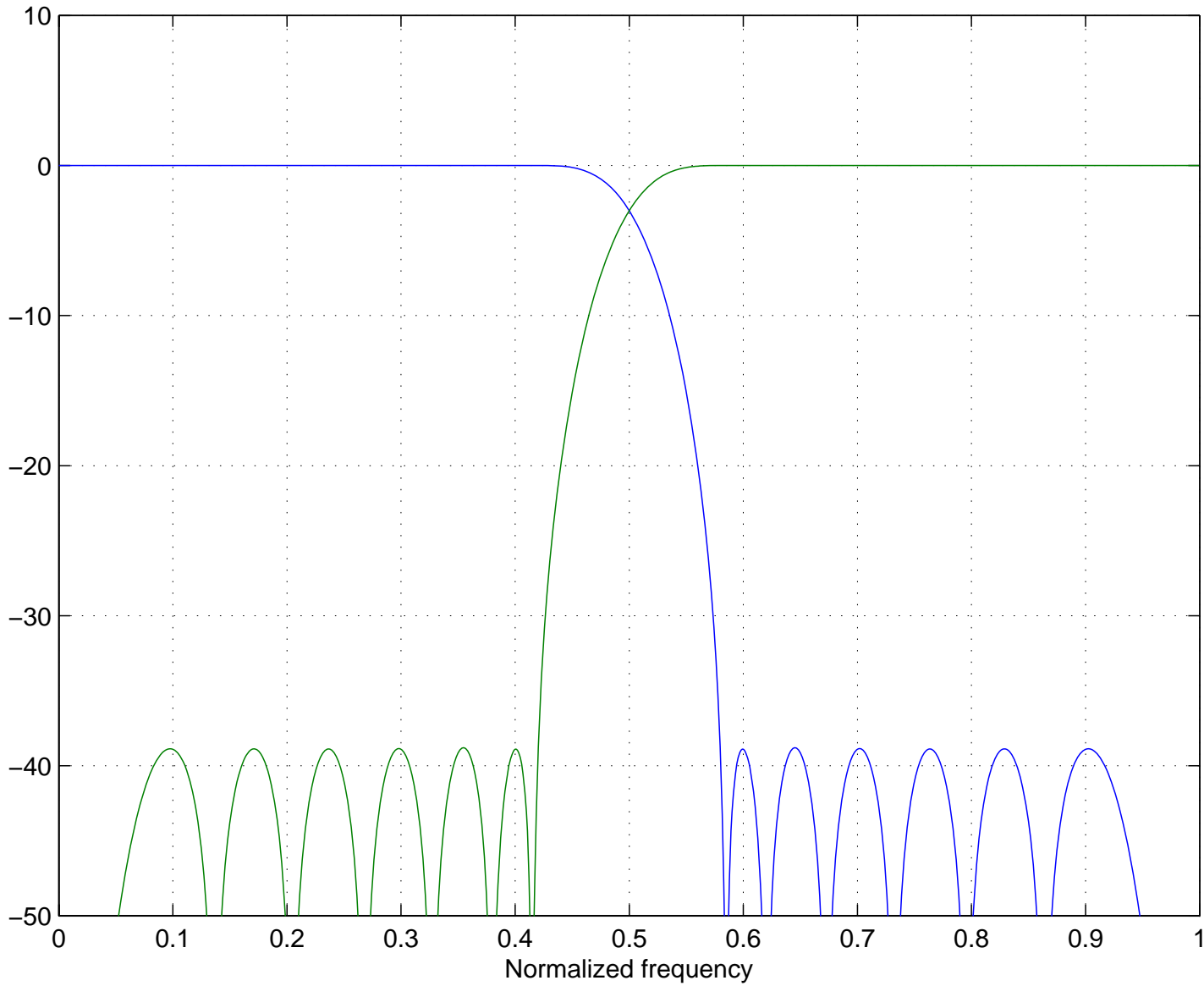


Table 5: Minimax with  $N = 64$ ,  $\omega_a = 0.57\pi$

$L$	Instantaneous Energy in Stopband	Largest Equation Error
0	$1.8886 \times 10^{-7}$	$5 \times 10^{-16}$
1	$2.0031 \times 10^{-7}$	$9.2014 \times 10^{-7}$
2	$2.0459 \times 10^{-7}$	$4.1258 \times 10^{-7}$
3	$2.0637 \times 10^{-7}$	$1.9405 \times 10^{-7}$
4	$2.1762 \times 10^{-7}$	$4.0643 \times 10^{-6}$
5	$2.4163 \times 10^{-7}$	$9.9786 \times 10^{-8}$

Minimax design with  $N = 64$ ,  $\omega_a = 0.57\pi$ , and  $L = 5$

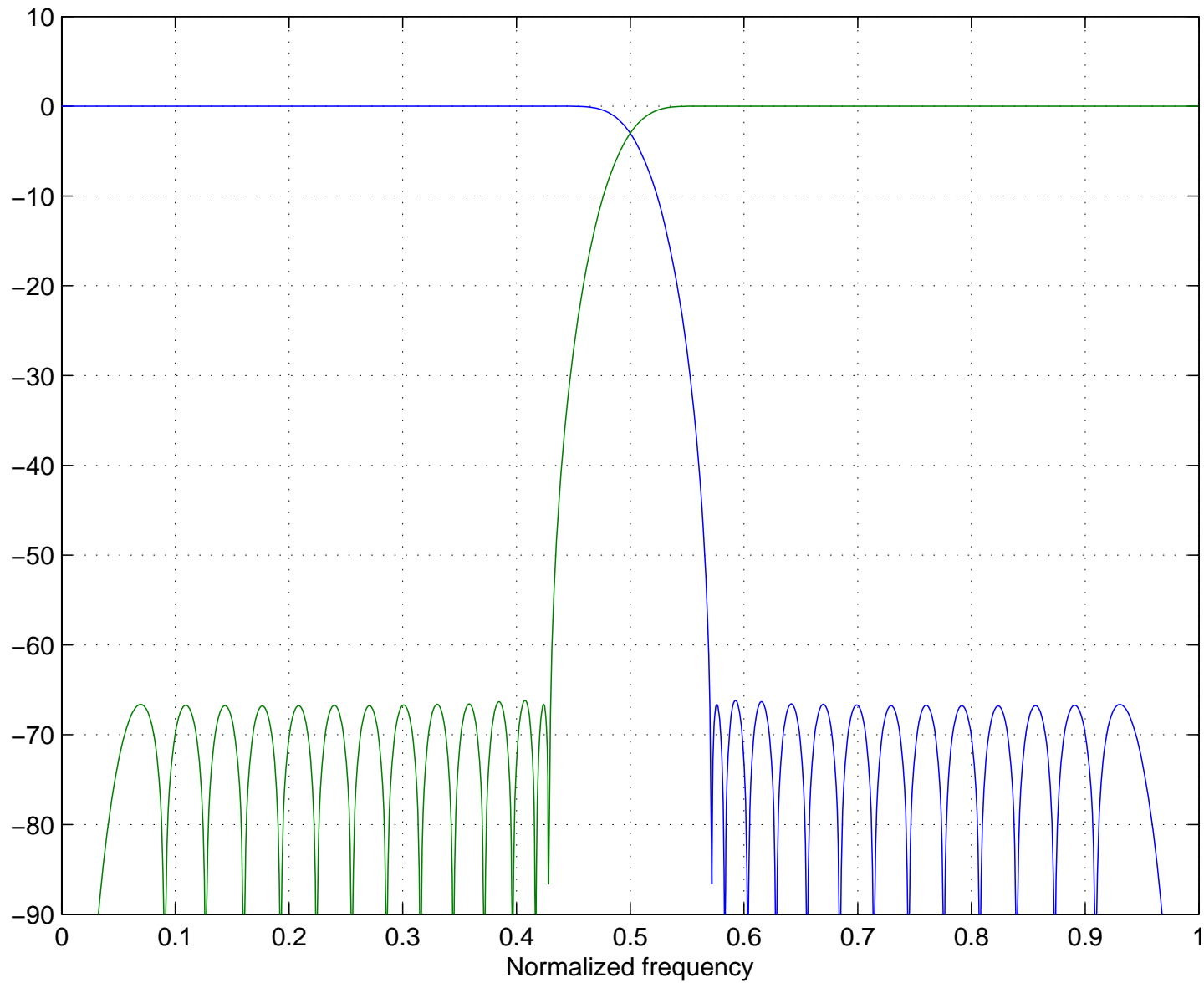


Table 6: Minimax with  $N = 96$ ,  $\omega_a = 0.56\pi$

$L$	Instantaneous Energy in Stopband	Largest Equation Error
0	$3.1336 \times 10^{-9}$	$5 \times 10^{-16}$
1	$3.0323 \times 10^{-9}$	$8.8247 \times 10^{-7}$
2	$3.0654 \times 10^{-9}$	$2.5128 \times 10^{-5}$
3	$3.4075 \times 10^{-9}$	$1.0654 \times 10^{-16}$
4	$3.1281 \times 10^{-9}$	$4.0553 \times 10^{-7}$
5	$3.7121 \times 10^{-9}$	$1.0982 \times 10^{-5}$

Minimax design with  $N = 96$ ,  $\omega_a = 0.56\pi$ , and  $L = 3$

